# Identity and Harmony revisited[*]

Stephen Read
University of St Andrews
email: slr@st-and.ac.uk

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### Abstract

The standard rules for identity do not seem to lie in harmony, where the elimination-rule is justified by the meaning given to a logical operator by its introduction-rule. However, harmonious rules can be given for identity, showing that identity is a logical notion even on the inferentialist conception. These new rules are shown to be sound and complete. The present paper omits §3 of the original published version [7], which responded to a mistaken objection by the referee, as was shown by Michael Kremer in [3].

## 1 Harmony

The inferentialist account of logic says that the meaning of a logical operator is given by the rules for its application. Prior ([5]) showed that a simple and straightforward interpretation of this account of logicality reduces to absurdity. For if 'tonk' has the meaning given by the rules Prior proposed for it, contradiction follows. Accordingly, a more subtle interpretation of inferentialism is needed. Such a proposal was put forward initially by Gentzen ([2]) in 1934 and elaborated by Prawitz (e.g., [4]). The meaning of a logical expression is given by the rules for the assertion of statements containing that expression (as designated component); these are its introduction-rules. The meaning so given justifies further rules for drawing inferences from such assertions; these are its elimination-rules:

> "The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequence of these definitions." ([2] p. 80)

For example, if the only ground for assertion of '$p$ tonk $q$' is given by Prior's rule:

$$\frac{p}{p \text{ tonk } q} \text{ tonk-I}$$

then Prior mis-stated the elimination-rule. It should read

$$\frac{p \text{ tonk } q \quad \overset{(p)}{\overset{\vdots}{r}}}{r} \text{ tonk-E}$$

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[*]A revised version of 'Identity and Harmony', *Analysis* 64 (April 2004), 111-19.

that is, given '$p$ tonk $q$', and a derivation of $r$ from $p$ (the ground for asserting '$p$ tonk $q$'), we can infer $r$, discharging the assumption $p$. We can state the rule more simply as follows:

$$\frac{p \text{ tonk } q}{p}$$

For if we may infer whatever, $r$, we can infer from $p$, we can infer $p$ and then proceed to infer $r$, that is, what we can infer from $p$. Prior's mistake was to give a rule

$$\frac{p \text{ tonk } q}{q}$$

which was not justified by the introduction-rule for 'tonk'.

A strong version of inferentialism says that an expression is a logical one if and only if its meaning can be (fully) characterized by such a (set of) introduction-rule(s). (See, e.g., [1] ch. 9 ff.) For example, '&' is a logical operator, for its meaning is given by

$$\frac{p \quad q}{p \text{ \& } q} \text{ \&I}$$

Suppose something, call it $r$, follows from $p$ & $q$. Something follows from $p$ & $q$ if and only if it follows from the grounds for the assertion of $p$ & $q$, that is, assertions of $p$ and of $q$. We can write this as follows:[1]

$$\frac{p \text{ \& } q \quad \overset{(p,q)}{\overset{\vdots}{r}}}{r} \text{ \&E}'$$

This gives the general form of the elimination-rule for '&', saying when something can be inferred from an assertion of $p$ & $q$. Again, it simplifies: rather than infer from $p$ & $q$ what we can infer from $p$ and $q$, we can simply infer $p$ and $q$ and then proceed to infer what we can from them each. Thus we obtain the standard rules of Simplification:

$$\frac{p \text{ \& } q}{p} \text{ \&E} \qquad\qquad \frac{p \text{ \& } q}{q} \text{ \&E}$$

Similarly, '$\rightarrow$' is a logical expression, for its meaning is given by the rule of Conditional Proof:

$$\frac{\overset{(p)}{\overset{\vdots}{q}}}{p \rightarrow q} \rightarrow \text{I}$$

Suppose something, call it $r$, follows from $p \rightarrow q$. Something follows from $p \rightarrow q$ if and only if it follows from the grounds for the assertion of $p \rightarrow q$, that is, a derivation of $q$ from $p$. Once again, if we may infer whatever we can infer from a derivation of $q$ from $p$, we can simply infer $q$ from $p$ and then proceed to infer whatever it is that we can infer from that inference. We can write this as follows:

$$\frac{p \rightarrow q \quad p}{q} \rightarrow \text{E}$$

Thus the elimination-rule for '$\rightarrow$', saying when something can be inferred from an assertion of $p \rightarrow q$, is the familiar rule of Modus Ponens. (I'll assume for the

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[1]For an improved account of the justificaation for the elimination-rules for & and other connectives, see [6].

purposes of this paper that '→' is material implication, but other interpretations of the conditional can result from interpreting the schemata →I and →E differently.)

On this account, logicality requires harmony, and harmony suffices for logicality. Rules are in harmony if the elimination-rules draw no more and no less from an assertion than the introduction-rules warrant. (See, e.g., [1] ch. 11, [8] ch. 9, [6].) Suitable analysis, similar to that above, shows that '∨', '¬', '∀', '∃', '□' and '◇' are all logical. Hence, an operator is logical iff there are introduction- and elimination-rules for the operator which lie in the appropriate harmony.

## 2 Identity

Is identity a logical operator? The rules for identity in a natural deduction setting are usually given in the form of Reflexivity and Congruence (see, e.g., [9] p. 77):

$$\overline{a = a} \ \text{Refl} \qquad\qquad \frac{a = b \quad p}{p(b/a)} \ \text{Congr}$$

Here, $p(b/a)$ denotes the result of replacing one or more occurrences of the term $a$ in $p$ by $b$. Refl would seem to be an introduction-rule for '='. It gives the ground for asserting formulae containing '=' as designated constituent. But it clearly does not justify Congr. Does it follow that '=' is not a logical operator? If the standard rules for '=' are not in harmony, strong inferentialism will entail that '=' is not logical only if there is no account of '=' using rules which are in harmony. Is there such an account of identity? What introduction-rule would justify =E?

The problem with the standard rules for '=' is that Refl seems too weak to justify Congr. So an idea for a harmonious theory of identity is simple: since the elimination-rule for '=' is the indiscernibility of identicals, the ground for asserting an identity must be the identity of indiscernibles. For if the introduction-rule for '=' were the identity of indiscernibles, and this was the only ground for asserting identity, Gentzen's proposal would justify as elimination-rule the indiscernibility of identicals. Here are the details.

In second-order logic, we can define identity:

$$a = b =_{df} \forall F(Fa \leftrightarrow Fb).$$

(Here $p \leftrightarrow q =_{df} (p \to q) \mathbin{\&} (q \to p)$.) In first-order logic, we cannot express the quantification over properties directly. However, even at first-order, we can express the indiscernibility of $a$ and $b$ by requiring a proof of $Fa \leftrightarrow Fb$ for arbitrary $F$, that is, where '$F$' (as a predicate variable) does not occur free—that is, does not occur—in any undischarged assumption. Moreover, we can simplify the premise to an implication, $Fa \to Fb$. For if this is provable for arbitrary $F$, we can show that $Fb \to Fa$ holds for arbitrary $F$ too. For suppose $Fb$ and $\neg Fa$. Replace '$F$' throughout the proof of $Fa \to Fb$ by '$\neg F$', permissible since '$F$' does not occur in any assumption. We obtain $\neg Fa \to \neg Fb$, whence $\neg Fb$, contradicting our hypothesis. So by classical *reductio*, $Fa$, that is, $Fb \to Fa$, where again, '$F$' is not free in any assumption.

We can eliminate the occurrence of '→' here by replacing the conditional $Fa \to Fb$ by a derivation of $Fb$ from $Fa$, to keep the rule pure (that is, with explicit reference only to the connective in question).

Thus we can formulate our rule of =I in first-order as follows:

$$
\begin{array}{c}
(Fa) \\
\vdots \\
\dfrac{Fb}{a = b} \ \text{= I}
\end{array}
$$

provided '$F$' does not occur (as a predicate variable) in any assumption other than $Fa$. The usual reflexivity axiom, $a = a$, follows immediately, since $Fa \vdash Fa$ is trivial. '$F$' does not occur in any auxiliary assumption, since there are none.

How does Gentzen's remark show that the indiscernibility of identicals is justified by =I, the identity of indiscernibles? Intuitively, the justification is immediate, for if all that $a = b$ says is that $a$ and $b$ are indiscernible (as shown by the form of =I), then if $a = b$ is true, $a$ and $b$ must be indiscernible. But to spell this out in detail is harder. We proceed in two stages. First, we show that =I justifies the special case of =E (that is, Congr) where $A(a)$ is an atomic predication $Fa$. Recall how Modus Ponens ($\rightarrow$E) was justified by $\rightarrow$I in §1. This time suppose something follows from $a = b$. Something follows from $a = b$ if and only if it follows from the grounds for the assertion of $a = b$, that is, from a derivation of $Fb$ from $Fa$ for some predicate variable $F$. Again, if we may infer (from $a = b$) whatever we can infer from a derivation of $Fb$ from $Fa$, we can simply infer $Fb$ from $Fa$ and then proceed to infer whatever it is that we can infer from that inference. We can write this as follows:

$$
\dfrac{a = b \quad Fa}{Fb} \ \text{= E}'
$$

But Congr (the standard version of =E) is formulated for an arbitrary context, $A(a)$:

$$
\dfrac{a = b \quad A(a)}{A(b)} \ \text{= E}
$$

So the second stage of our justification of =E is to obtain it from =E$'$ by an application of second-order $\exists$E applied to $F$:

$$
\dfrac{
\dfrac{
\dfrac{A(a)}{A(a) \ \& \ \forall x(A(x) \leftrightarrow A(x))}
}{\exists F(Fa \ \& \ \forall x(Fx \leftrightarrow A(x)))} \ *
\quad
\dfrac{a = b \quad \dfrac{\dfrac{Fa \ \& \ \forall x(Fx \leftrightarrow A(x))^1}{Fa}}{Fb} \quad \dfrac{\dfrac{Fa \ \& \ \forall x(Fx \leftrightarrow A(x))^1}{\forall x(Fx \leftrightarrow A(x))}}{Fb \leftrightarrow A(b)}}{A(b)}
}{A(b)} \ \exists \text{E(1)}
$$

Note that the step marked (*) is a strong form of (second-order) $\exists$I equivalent to second-order Comprehension, or $\lambda$-conversion.

# 3  Soundness[2]

We now show that =I is sound, that is, that if $X, Fa \models Fb$, where '$F$' does not occur in $X$, then $X \models a = b$. Suppose that $X, Fa \models Fb$, that is, $Fb$ is true in every model in which every member of $X$ and $Fa$ are true. Consider an interpretation $I$ in which every member of $X$ and $Fa$ are true. Suppose $I(a) = d$ and let $I'$ differ from $I$ at most in the interpretation of $F$, such that

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[2]§3 of the original paper attempted to establish soundness by introducing the concept of Leibnizian models, in response to an objection by the referee. The response was shown to be unnecessary and confused in [3]. The following proof follows Kremer's proof closely.

the only member of $I(F)$ is $d$. Since $F$ does not occur in any member of $X$, and every member of $X$ is true in $I$, every member of $X$ is also true in $I'$. Moreover, $I'(a) = I(a) = d \in I'(F)$, so $Fa$ is true under $I'$. So $Fb$ si true under $I'$, that is, $I'(b) \in I'(F)$, so $I(b) = I'(b) = d \; I'(a) = I(a)$, whence $a = b$ is true under $I$. Hence $X \models a = b$, as required.

# 4 Conclusion

The indiscernibility of identicals, at least in extensional contexts, is usually regarded as uncontentious, even definitive of identity—identity is a congruence, indeed, the universal congruence. The identity of indiscernibles, in contrast, is a highly contested, not to say, false thesis. Does it follow that the above harmonious account of identity is mistaken, in accepting the contentious thesis? The soundness of =I shows that this is not so. The account of identity given by =I is the standard one: =I is sound by standard semantics. Moreover, =I and =E lie in harmony, so '=' is indeed a logical expression.

# References

[1] M. Dummett, *Logical Basis of Metaphysics*. London: Duckworth, 1991.

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[3] M. Kremer, 'Read on identity and harmony—a friendly correction and simplification', *Analysis*, 67, April 2007, pp. 157-59.

[4] D. Prawitz, 'Meaning and proofs: on the conflict between intuitionistic and classical logic', *Theoria*, 43, 1977, pp. 2-40.

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