General-Elimination Harmony and Higher-Level Rules*

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Abstract

Michael Dummett introduced the notion of harmony in response to Arthur Prior's tonkish attack on the idea of proof-theoretic justification of logical laws (or analytic validity). But Dummett vacillated between different conceptions of harmony, in an attempt to use the idea to underpin his anti-realism. Dag Prawitz had already articulated an idea of Gerhard Gentzen's into a procedure whereby elimination-rules are in some sense functions of the corresponding introduction-rules. The resulting conception of general-elimination harmony ensures that the rules are transparent in the meaning they confer, in that the elimination-rules match the meaning the introduction-rules confer. The general-elimination rules which result may be of higher level, in that the assumptions discharged by the rule may be of (the existence of) derivations rather than just of formulae. In many cases, such higher-level rules may be "flattened" to rules discharging only formulae. However, such flattening is often only possible in the richer context of so-called "classical" or realist negation, or in a multiple-conclusion environment. In a constructivist context, the flattened rules are harmonious but not stable.

1 Analytic Validity

In a famous article published in December 1960, Arthur Prior (1960) attacked the idea that

> "there are inferences whose validity arises solely from the meanings of certain expressions occurring in them."

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He argued that validity must instead be based on truth-preservation, not on meaning. To illustrate what he considered to be the problem with what he dubbed analytic validity, Prior introduced a new connective 'tonk' with the rules:

$$\frac{\alpha}{\alpha \text{ tonk } \beta}\ \text{tonk-I} \qquad\qquad \frac{\alpha \text{ tonk } \beta}{\beta}\ \text{tonk-E}$$

By chaining together an application of tonk-I with one of tonk-E, we can apparently derive any proposition (β) from any other (α). This is clearly absurd and disastrous. How can one possibly *define* such an inference into existence?

We may agree with Prior that 'tonk' had not been given any recognisable meaning by these rules. Rather, whatever meaning tonk-introduction had conferred on the neologism 'tonk' was then contradicted by Prior's tonk-elimination rule. But we might respond to Prior by claiming that if rules were set down for a term which did properly confer meaning on it, then certain inferences would be "analytic" in virtue of that meaning. What constraints must rules satisfy in order to confer a coherent meaning on the terms involved?

Michael Dummett (1973) introduced the term 'harmony' for this constraint: in order for the rules to confer meaning on a term, two aspects of its use must be in harmony. Those two aspects are the grounds for an assertion as opposed to the consequences we are entitled to draw from such an assertion. Those whom Prior was criticising, Dummett claimed, had committed the "error" of failing to appreciate

> "the interplay between the different aspects of 'use', and the requirement of harmony between them. (Dummett, 1973, p. 396) If the linguistic system as a whole is to be coherent, there must be a harmony between these two aspects." (Dummett, 1991, p. 221)

In appealing to this connection between the grounds, or introduction-rules, and the consequences, or elimination-rules, Dummett was following the lead of Dag Prawitz, who in turn was following out an idea of Gerhard Gentzen's, in a famous and much-quoted passage where he says that

> "the E-inferences are, through certain conditions, *unique* consequences of the respective I-inferences." (Gentzen, 1969, p. 81)

In a series of articles on the "foundations of a general proof theory" published in the early 1970s, Prawitz (1973, 1974, 1975) set out to find a characterization of validity of argument independent of model theory, as typified by Tarski's truth-preservationist account of logical consequence. Following Gentzen's idea in the passage cited above, Prawitz accounts an argument or derivation valid by virtue of the meaning or definition of the logical constants encapsulated in the I-rules. Suppose we take the I-rules as given. Then any argument (or in the general case, argument-schema) is valid if there is a "justifying operation" ultimately articulating the argument into the application of I-rules to atomic sentences:[1]

> "The main idea is this: while the introduction inferences represent the form of proofs of compound formulas by the very meaning of the logical constants ... and hence preserve validity, other inferences have to be justified by the evidence of operations of a certain kind." (Prawitz, 1973, p. 234)

What Prawitz does, in fact, is frame his E-rules in such a way that such a justification is possible. Given a set of I-rules for a connective (in general, there may be several, as in the familiar case of '$\vee$'), the E-rules (again, there may be several, as in the case of '$\wedge$') which are justified by the meaning so conferred are those which will permit an operation of Prawitz' kind. The principle underlying this procedure is called by Prawitz (1965), following Paul Lorenzen, the "inversion principle". Prawitz refers to Lorenzen (1955), and more particularly to Hermes (1959, p. 65), for the full statement of the principle. Prawitz (1965, p. 33) writes:[2]

> "Let α be an application of an elimination rule that has B as consequence. Then, deductions that satisfy the sufficient condition [...] for deriving the major premiss of α, when combined with deductions of the minor premisses of α (if any), already 'contain' a deduction of B; the deduction of B is thus obtainable directly from the given deductions without the addition of α."

Each E-rule is harmoniously justified by satisfying the constraint that whenever its premises are provable (by application of one of the I-rules), the conclusion is derivable (by use of the assertion-conditions framed in the I-rule), that is, it is admissible (*zulässig*: Lorenzen 1955, p. 30; Hermes 1959, p. 63). Francez and Dyckhoff (2012) introduced the term "General-Elimination Harmony" for the form which this procedure accords to the E-rules.[3]

2 General-Elimination Harmony

Suppose there are m I-rules for a connective 'δ', each with n_i premises, $0 \leq i \leq m$:[4]

$$\frac{\pi_{i1} \quad \dots \quad \pi_{in_i}}{\delta\vec{\alpha}}\ \delta\text{-I}_i$$

Here $\delta\vec{\alpha}$ is a formula with main connective 'δ'. Each π_{ij}, $0 \leq j \leq n_i$, may be a wff (as in $\wedge$I), or a derivation of a wff from certain assumptions which are

[1] See also Prawitz (2006) and Read (2014).

[2] Cf. Schroeder-Heister (2006), (2007).

[3] For an extended discussion of GE-harmony, see Read (2010).

[4] m may be zero, as Prawitz (1973, p. 243) notes is the case for the absurdity constant, $\perp$, which has no grounds for its assertion.

discharged by the rule (as in →I). In accordance with the inversion principle, this set of I-rules justifies $\prod_{i=1}^{m} n_i$ E-rules, each of the form:[5]

$$\dfrac{\delta\vec{\alpha} \quad \begin{matrix}[\pi_{1j_1}] \\ \vdots \\ \gamma\end{matrix} \quad \cdots \quad \begin{matrix}[\pi_{mj_m}] \\ \vdots \\ \gamma\end{matrix}}{\gamma}\ \delta\text{-E}$$

Each minor premise derives γ from one of the grounds, π_{ij_i}, in the i-th rule for asserting $\delta\vec{\alpha}$.

The justification is this: the GE-procedure ensures that one can infer γ from $\delta\vec{\alpha}$ whenever one can infer γ from one of the grounds for assertion of $\delta\vec{\alpha}$. Consequently, the actual assertion of $\delta\vec{\alpha}$ is an unnecessary detour:

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \pi_{i1}\end{matrix} \quad \cdots \quad \begin{matrix}\vdots \\ \pi_{in_i}\end{matrix}}{\delta\vec{\alpha}}\ \delta\text{-I} \quad \begin{matrix}[\pi_{1j_1}] \\ \vdots \\ \gamma\end{matrix} \quad \cdots \quad \begin{matrix}[\pi_{mj_m}] \\ \vdots \\ \gamma\end{matrix}}{\gamma}\ \delta\text{-E}$$

converts to

$$\begin{matrix}\vdots \\ \pi_{ij_i} \\ \vdots \\ \gamma\end{matrix}$$

Having one minor premise in each E-rule drawn from among the premises for each I-rule ensures that, whichever I-rule justified assertion of $\delta\vec{\alpha}$ (here it was the i-th), one of its premises can be paired with one of the minor premises to remove the unnecessary application of δ-I immediately followed by δ-E.

The idea behind harmony is that the elimination-rule should allow one to infer all and only what is justified by the meaning conferred by the introduction-rule. The above procedure certainly shows that the E-rule permits one to infer no more than is so justified. But it does not show that the rule permits inference of everything that is justified by the meaning so conferred. The idea that the E-rule should not only not be too weak but also not too strong was called by Dummett (1991, ch. 13), "stability". For example, consider the Curry-Fitch rules for $\Diamond$ (possibility):[6]

$$\dfrac{\alpha}{\Diamond\alpha}\ \Diamond\text{-I} \qquad \text{and} \qquad \dfrac{\Diamond\alpha \quad \begin{matrix}[\alpha] \\ \vdots \\ \gamma\end{matrix}}{\gamma}\ \Diamond\text{-E}$$

[5]If $m = 0$, the empty product predicts one E-rule, to infer an arbitrary conclusion from $\bot$. If $n_i = 0$ for some i, the product is 0. E.g., if we introduce $\top$ by an I-rule with no premises (even if we give alternative, more restrictive, grounds for its assertion), $\top$ is a tautology, and nothing can be inferred from it which is not already provable.

[6]See Curry (1950, ch. V), Fitch (1952, ch. 3), Prawitz (1965, ch. VI).

provided that in the case of $\Diamond$-E, every assumption on which the minor premise γ depends, apart from α (the so-called parametric formulae), is modal (that is, has the form $\Box\beta$) and γ is co-modal (that is, has the form $\Diamond\beta$). These rules are not stable: the (unrestricted) rule $\Diamond$-I does not justify the restriction put on $\Diamond$-E. $\Diamond$-I appears to say that $\Diamond\alpha$ just means α—that is, $\Diamond\alpha$ is assertible just when α is. But the model theory shows that the rules do define possibility. Quite how they interact to do so is far from obvious.[7]

Dummett and Prawitz (and others) make a yet stronger claim: that an inference is not justified if the rules are not harmonious.[8] For example, Dummett (1991, p. 299) claims that classical logic, with classical negation, is incoherent since it cannot be given harmonious rules. In my view, this asks too much of harmony and the constraints on the rules it invokes. An example (Dummett, 1991, p. 291) is the minimal theory of negation. Dummett's introduction-rule for negation:

$$\frac{\begin{matrix}[\alpha]\\ \vdots \\ \neg\alpha\end{matrix}}{\neg\alpha}\ \neg\text{-I}$$

does not justify the intuitionistically valid elimination-rule:

$$\frac{\neg\alpha \quad \alpha}{\beta}\ \text{EFQ}$$

Nonetheless, the intuitionistic E-rule is still valid, and between them the two rules give the intuitionistic theory of negation. EFQ simply is not justified by the I-rule which Dummett proposes. To discern the meaning which these rules define it is not enough just to look at the I-rule. One must look at the elimination-rule too, just as one had to do with the inharmonious rules for $\Diamond$. What harmony and stability can do for us is ensure that the I- and E-rules confer the same meaning, and so ensure that the meaning is transparent in the grounds for assertion, that is, the I-rule. We can see this by considering some particular cases.

Here is an example, perhaps over-familiar, but suitably revealing. Given as I-rule:

$$\frac{\alpha \qquad \beta}{\alpha \wedge \beta}\ \wedge\text{I}$$

the inversion principle yields two generalized $\wedge$-E rules, assuming $\wedge$-I to

[7]To obtain harmonious rules, the right response is, of course, not to strengthen $\Diamond$-E to match $\Diamond$-I, but to find some way to weaken $\Diamond$I. For one possible solution, see Read (2008).

[8]See, e.g., Dummett (1991, pp. 286-7), qualified only by the remark: "when [the] rules are held completely to determine the meanings of the logical constants." Cf. Prawitz (1985, p. 138), Schroeder-Heister (2006, p. 532), Tennant (nd, §11).

exhaust the grounds for asserting $\alpha \wedge \beta$ (so $m = 1$ and $n_1 = 2$):

$$\dfrac{\alpha \wedge \beta \quad \begin{matrix}[\alpha] \\ \vdots \\ \gamma\end{matrix}}{\gamma}\ \wedge\text{-E}_1 \qquad \text{and} \qquad \dfrac{\alpha \wedge \beta \quad \begin{matrix}[\beta] \\ \vdots \\ \gamma\end{matrix}}{\gamma}\ \wedge\text{-E}_2$$

The generalized $\wedge$-E rules yield the more familiar $\wedge$-E rules of Simp(lification) immediately as instances, by letting γ be α and β respectively:

$$\dfrac{\alpha \wedge \beta \quad \begin{matrix}[\alpha] \\ \vdots \\ \alpha\end{matrix}}{\alpha}\ \wedge\text{-E}_1 \qquad \text{and} \qquad \dfrac{\alpha \wedge \beta \quad \begin{matrix}[\beta] \\ \vdots \\ \beta\end{matrix}}{\beta}\ \wedge\text{-E}_2$$

which reduce to

$$\dfrac{\alpha \wedge \beta}{\alpha}\ \text{Simp}_1 \qquad \text{and} \qquad \dfrac{\alpha \wedge \beta}{\beta}\ \text{Simp}_2$$

given that we can always derive γ from γ, for all γ. Conversely, $\wedge$-E$_1$ follows from Simp$_1$, that is, that if there is a derivation of γ from α, then γ follows from $\alpha \wedge \beta$:

$$\begin{matrix}\dfrac{\alpha \wedge \beta}{\alpha} \\ \vdots \\ \gamma\end{matrix}$$

and the same for $\wedge$-E$_2$.

Dyckhoff and Francez, Schroeder-Heister and others, have a single form of the generalized rule:[9]

$$\dfrac{\alpha \wedge \beta \quad \begin{matrix}\underbrace{[\alpha] \quad [\beta]} \\ \vdots \\ \gamma\end{matrix}}{\gamma}\ \wedge\text{-GE}$$

To see that $\wedge$-GE is equivalent to the conjunction of $\wedge$-E$_1$ and $\wedge$-E$_2$, let us replace the two-dimensional representation of the derivation of γ from α and β by the linear form $\alpha, \beta \Rightarrow \gamma$. Then we can derive each of $\wedge$-E$_1$ and $\wedge$-E$_2$ from $\wedge$-GE:

$$\dfrac{\alpha \wedge \beta \quad \dfrac{\alpha \Rightarrow \gamma}{\alpha, \beta \Rightarrow \gamma}\ \text{K (Weakening)}}{\gamma}\ \wedge\text{-GE}$$

and the same for β. Conversely,

$$\dfrac{\alpha \wedge \beta \quad \dfrac{\alpha \wedge \beta \quad \alpha, \beta \Rightarrow \gamma}{\beta \Rightarrow \gamma}\ \wedge\text{-E}_1}{\gamma}\ \wedge\text{-E}_2$$

[9]See, e.g., Francez and Dyckhoff (2012, p. 615), Schroeder-Heister (1984, p. 1294), Negri and von Plato (2001, p. 7).

What this shows is $\alpha \wedge \beta, \alpha \wedge \beta \Rightarrow \gamma$, and $\wedge$-GE follows by Contraction (W).

Thus we have two competing forms of $\wedge$-E, though they are equivalent, given Contraction and Weakening. But in the absence of W and K, which is the right form? Recall the additive and multiplicative left-rules for $\wedge$ and $\otimes$ in linear logic:[10]

$$\frac{\alpha, \Gamma \Rightarrow \Theta}{\alpha \wedge \beta, \Gamma \Rightarrow \Theta} \wedge\Rightarrow \qquad \frac{\beta, \Gamma \Rightarrow \Theta}{\alpha \wedge \beta, \Gamma \Rightarrow \Theta} \wedge\Rightarrow \qquad \frac{\alpha, \beta, \Gamma \Rightarrow \Theta}{\alpha \otimes \beta, \Gamma \Rightarrow \Theta} \otimes\Rightarrow$$

Clearly, $\wedge$-GE gives the multiplicative E-rule for $\otimes$, whereas $\wedge$-E$_1$ and $\wedge$-E$_2$ give the correct E-rules for additive $\wedge$. In the presence of W and K, the additive/multiplicative distinction is erased, but to give the rules in their proper form, we need to give separate E-rules for $\wedge$, each corresponding to one of the premises in $\wedge$-I:

$$\frac{\alpha \wedge \beta \quad \alpha \Rightarrow \gamma}{\gamma} \wedge\text{-E}_1 \qquad \text{and} \qquad \frac{\alpha \wedge \beta \quad \beta \Rightarrow \gamma}{\gamma} \wedge\text{-E}_2$$

confirming the correctness of the GE-procedure.

$\wedge$-I, $\wedge$-E$_1$ and $\wedge$-E$_2$ are harmonious. But are they stable? That is, do $\wedge$-E$_1$ and $\wedge$-E$_2$ allow one to derive all the consequences that the meaning encapsulated in $\wedge$-I justifies? Davies and Pfenning (2001, p. 560) call harmony, "local soundness", that the E-rules allow one to derive no more than the I-rule justifies; that they allow one to derive no less, they dub "local completeness". They write:

> "Local completeness ensures that we can recover all information present in a connective: there is some way to apply the elimination rules so we can reconstitute a proof of the original proposition using its introduction rules."

But as Dummett (1991, pp. 288-9) showed, the restricted $\vee$-rules of quantum logic satisfy that condition:

$$\frac{\alpha \vee \beta \qquad \dfrac{[\alpha]^1}{\alpha \vee \beta} \vee\text{I}_1 \qquad \dfrac{[\beta]^2}{\alpha \vee \beta} \vee\text{I}_2}{\alpha \vee \beta} \vee\text{E}_{\text{Q}}(1,2)$$

where no undischarged assumptions are allowed in the minor premises of $\vee$E$_{\text{Q}}$. Yet the quantum $\vee$-rules are clearly incomplete, in that they do not allow us to prove the distribution of '$\wedge$' over '$\vee$'.

The test is too weak—we need to recover not just the original proposition, but its grounds. But how can we recover the original grounds, *viz* α or β, from $\alpha \vee \beta$? In multiple-conclusion reasoning, it is straightforward. We need only derive the sequent $\alpha \vee \beta \Rightarrow \alpha, \beta$. In single-conclusion systems, we can at best show that $\neg\alpha, \alpha \vee \beta \vdash \beta$ and $\neg\beta, \alpha \vee \beta \vdash \alpha$, that is, that if one of the grounds for asserting $\alpha \vee \beta$ fails, then, given that $\alpha \vee \beta$ holds, the other

[10] See, e.g., Girard et al. (1989, p. 152).

ground must hold. In general, we can show that we can derive π_{kl} from $\delta\vec{\alpha}$ for every $k \leq m$ and $l \leq n_k$, given the falsity of π_{ij} for each $i \neq k$ and some $j \leq n_i$.[11] The general-elimination rules are not only in harmony with the I-rules, they are moreover, stable.

3 Flattening the Rules

As a second example, consider the I-rule for implication:

$$\dfrac{\begin{matrix}[\alpha]\\ \vdots \\ \beta\end{matrix}}{\alpha \to \beta} \to\text{-I} \qquad \text{that is,} \qquad \dfrac{\alpha \Rightarrow \beta}{\alpha \to \beta} \to\text{-I}$$

inferring (an assertion of the form) $\alpha \to \beta$ from (a derivation of) β, permitting the discharge of (zero or more occurrences of) α. Whatever form $\to$-E has, there must be the appropriate justificatory operation of which Prawitz spoke. That is, we should be able to infer from an assertion of $\alpha \to \beta$ no more (and no less) than we could infer from whatever warranted assertion of $\alpha \to \beta$. We can represent this as follows:

$$\dfrac{\alpha \to \beta \qquad \begin{matrix}\left[\begin{matrix}[\alpha]\\ \vdots \\ \beta\end{matrix}\right]\\ \vdots \\ \gamma\end{matrix}}{\gamma} \to\text{-E}^{12} \qquad \text{that is,} \qquad \dfrac{\alpha \to \beta \qquad \begin{matrix}[\alpha \Rightarrow \beta]\\ \vdots \\ \gamma\end{matrix}}{\gamma} \to\text{-E}^{13}$$

Thus, if we can infer γ from assuming the existence of a derivation of β from α, we can infer γ from $\alpha \to \beta$.

What does

$$\begin{matrix}[\alpha \Rightarrow \beta]\\ \vdots \\ \gamma\end{matrix} \qquad \text{that is,} \qquad \begin{matrix}\left[\begin{matrix}[\alpha]\\ \vdots \\ \beta\end{matrix}\right]\\ \vdots \\ \gamma\end{matrix}$$

mean? It says that we have a derivation of γ on the assumption that we have a derivation of β from α. Hence, if we have a derivation of α, we may

[11] See Jacinto and Read (nd).

[12] The form of representation here is inspired by Gentzen's notation in the draft of his dissertation, Gentzen (1932).

[13] Note that in $\alpha \Rightarrow \beta$, the assumption α is closed, either by a rule discharging the assumption (e.g., $\to$-I) or by a derivation of α (e.g., in the proof of the minor premise of $\to$-E).

assume we are able to derive β, from which we derive γ. That is,

$$\dfrac{\alpha \to \beta \qquad \begin{array}{c}\left[\begin{array}{c}[\alpha] \\ \vdots \\ \beta\end{array}\right] \\ \mathcal{D} \\ \gamma\end{array}}{\gamma} \to\text{-E}$$

means that we can connect a derivation of α with a derivation of γ from β as follows:

$$\dfrac{\alpha \to \beta \qquad \begin{array}{c}\mathcal{D}' \\ \left[\begin{array}{c}\alpha \\ \vdots \\ \beta\end{array}\right] \\ \mathcal{D}'' \\ \gamma\end{array}}{\gamma} \to\text{-E}$$

$\to$-E is an example of what Schroeder-Heister (1984) called "higher-level" rules, where what is assumed (and discharged) is a rule (here the inference of β from α) rather than just a formula. We can "flatten" the rule by separating the derivation $\mathcal{D}'$ of α from the derivation $\mathcal{D}''$ of γ from β:[14]

$$\dfrac{\alpha \to \beta \quad \begin{array}{c}\mathcal{D}' \\ \alpha\end{array} \quad \begin{array}{c}[\beta] \\ \mathcal{D}'' \\ \gamma\end{array}}{\gamma} \to\text{-E}'$$

There are now no "higher-level" assumptions, just minor premises α and γ, where in $\to$-E$'$ any assumption of the form β used to derive γ may be discharged.

Another way to think of this move appeals to the sequent calculus formulation, as before. The minor premise of $\to$-E reads: $(\alpha \Rightarrow \beta) \Rightarrow \gamma$. Using Gentzen's $\Rightarrow$-left rule, we have

$$\dfrac{\alpha \to \beta \quad \dfrac{\Rightarrow \alpha \quad \beta \Rightarrow \gamma}{(\alpha \Rightarrow \beta) \Rightarrow \gamma} \Rightarrow\text{-left}}{\gamma} \to\text{-E}$$

Thus our generalized E-rule for $\to$ reads:

$$\dfrac{\alpha \to \beta \qquad \alpha \qquad \begin{array}{c}[\beta] \\ \vdots \\ \gamma\end{array}}{\gamma} \to\text{-E}'$$

Letting $\gamma = \beta$ and again assuming we can derive β from itself, we obtain the familiar rule of Modus Ponendo Ponens (MPP):

$$\dfrac{\alpha \to \beta \qquad \alpha}{\beta} \text{MPP}$$

Conversely, given the premises of $\to$-E$'$, we can derive γ using MPP:

$$\begin{array}{c}\dfrac{\alpha \to \beta \quad \alpha}{\beta} \text{MPP} \\ \vdots \\ \gamma\end{array}$$

[14] Dyckhoff (1988) was possibly the first to propose this formulation, which we can also find in, e.g., von Plato (2001, p. 545). Dyckhoff rejected it for reasons summarized in Dyckhoff (2013).

Similar considerations arise in the obvious introduction rule for equivalence $\leftrightarrow$, which requires both that β be derivable from α and vice versa:

$$\frac{\begin{array}{cc}[\alpha] & [\beta] \\ \vdots & \vdots \\ \beta & \alpha\end{array}}{\alpha \leftrightarrow \beta}\ \leftrightarrow\text{-I}$$

Here $m = 1$ and $n_1 = 2$, so there are two E-rules each with one minor premise:

$$\frac{\begin{array}{cc} & [\alpha \Rightarrow \beta] \\ & \vdots \\ \alpha \leftrightarrow \beta & \gamma\end{array}}{\gamma}\ \leftrightarrow\text{-E}_1 \qquad \frac{\begin{array}{cc} & [\alpha \Rightarrow \beta] \\ & \vdots \\ \alpha \leftrightarrow \beta & \gamma\end{array}}{\gamma}\ \leftrightarrow\text{-E}_2$$

Each simplifies by flattening of the rules and moves similar to those with the generalized rule for $\rightarrow$-E, to obtain:

$$\frac{\alpha \leftrightarrow \beta \quad \alpha}{\beta}\ \leftrightarrow\text{-E}_1' \qquad \frac{\alpha \leftrightarrow \beta \quad \beta}{\alpha}\ \leftrightarrow\text{-E}_2'$$

Suppose we now introduce a novel connective which disjoins the grounds for asserting $\alpha \leftrightarrow \beta$ instead of conjoining them:

$$\frac{\begin{array}{c}[\alpha] \\ \vdots \\ \beta\end{array}}{\alpha \odot \beta}\ \odot\text{-I}_1 \qquad \frac{\begin{array}{c}[\beta] \\ \vdots \\ \alpha\end{array}}{\alpha \odot \beta}\ \odot\text{-I}_2$$

Here we have two I-rules each with one premise ($m = 2$ and $n_1 = n_2 = 1$), so there will be one E-rule with two minor premises:

$$\frac{\begin{array}{ccc} & [\alpha \Rightarrow \beta] & [\beta \Rightarrow \alpha] \\ & \vdots & \vdots \\ \alpha \odot \beta & \gamma & \gamma\end{array}}{\gamma}\ \odot\text{-E}$$

Flattening of the rules yields:

$$\frac{\begin{array}{ccccc} & & [\beta] & & [\alpha] \\ & & \vdots & & \vdots \\ \alpha \odot \beta & \alpha & \gamma & \beta & \gamma\end{array}}{\gamma}\ \odot\text{-E}'$$

and the major premise seems redundant. We can prove γ directly from the minor premises (indeed, twice over).

This may seem puzzling, but in fact, reflection shows that it is to be expected, at least from a classical perspective. The two cases of $\odot$-I show that $\alpha \odot \beta$ means that either β is derivable from α (possibly given other assumptions) or α is derivable from β. That is a classical tautology: $(\alpha \to \beta) \vee (\beta \to \alpha)$. Take the following intuitionistic negation rules:

$$\dfrac{\begin{matrix}[\alpha]\\ \vdots\\ \beta\end{matrix} \qquad \begin{matrix}[\alpha]\\ \vdots\\ \neg\beta\end{matrix}}{\neg\alpha}\,\mathcal{R} \qquad\qquad \dfrac{\neg\alpha \qquad \alpha}{\beta}\,\mathcal{V}$$

(These are Gentzen's negation rules $\mathcal{R}$ and $\mathcal{V}$, which we will look at in the next section.) With them, we can prove $\neg\neg(\alpha \odot \beta)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(\alpha\odot\beta)}\,(1) \quad \dfrac{\dfrac{}{\alpha}\,(2)}{\alpha\odot\beta}\,\odot\text{-I}_2(3)}{\beta}\,\mathcal{V}}{\alpha\odot\beta}\,\odot\text{-I}_1(2) \qquad \dfrac{}{\neg(\alpha\odot\beta)}\,(1)}{\neg\neg(\alpha\odot\beta)}\,\mathcal{R}(1)$$

This will yield a classical proof of $\alpha \odot \beta$ by extending the above proof by an application of DN (double-negation elimination), or replacing the application of $\mathcal{R}$ by classical *reductio*:

$$\dfrac{\begin{matrix}[\neg\alpha]\\ \vdots\\ \beta\end{matrix} \qquad \begin{matrix}[\neg\alpha]\\ \vdots\\ \neg\beta\end{matrix}}{\alpha}\,\mathcal{CR}$$

The example of $\odot$ illustrates a general problem affecting the flattening procedure. In the case of $\to$ and $\leftrightarrow$, the flattened rule is easily shown to be as strong as the higher-level rule. But in the case of $\odot$ and other connectives, this is not true. The flattened rules, though harmonious, are not in general stable. Schroeder-Heister (nd) raises the issue, identifying two kinds of problem case. Take a case of δ-E:

$$\dfrac{\delta\vec{\alpha} \quad \begin{matrix}[\pi_{1j_1}]\\ \mathcal{D}_1\\ \gamma\end{matrix} \quad \cdots \quad \begin{matrix}[\beta_1 \Rightarrow \beta_2]\\ \mathcal{D}_i\\ \gamma\end{matrix} \quad \cdots \quad \begin{matrix}[\pi_{mj_m}]\\ \mathcal{D}_m\\ \gamma\end{matrix}}{\gamma}\,\delta\text{-E}$$

where discharged assumption π_{ij_i} is of higher level, assuming a derivation of β_2 from β_1. First, in any application, the assumption may have been discharged vacuously; that is, there may be a proof of γ not depending on the assumption of a derivation of β_2 from β_1 at all. Secondly, in the derivation of γ from the higher-level assumption of a derivation of β_2 from β_1, there may be some assumption made in the derivation of β_1 which is only discharged subsequent to the use of the higher-level assumption:

$$\begin{matrix}\mathcal{D}_i'\\ \beta_1\\ \vdots\\ \beta_2\\ \mathcal{D}_i''\\ \gamma\end{matrix}$$

That is, β_1 may depend on some assumption ϵ on which β_2, but not γ, also depends. So ϵ is discharged in the course of $\mathcal{D}''_i$. In a derivation using the flattened rule, however, ϵ is left undischarged in $\mathcal{D}'_i$ and is not available for use in $\mathcal{D}''_i$, which consequently is no longer a derivation.

We can see the problem plainly if we apply Davies and Pfenning's test for local completeness to $\alpha \odot \beta$. With the higher-level E-rule, a detour is easily introduced:

$$\dfrac{\alpha \odot \beta \qquad \dfrac{[\alpha \Rightarrow \beta]}{\alpha \odot \beta}\,\odot\text{-I}_1 \qquad \dfrac{[\beta \Rightarrow \alpha]}{\alpha \odot \beta}\,\odot\text{-I}_2}{\alpha \odot \beta}\,\odot\text{-E}$$

But with the flattened E-rule, the detour cannot be achieved:

$$\dfrac{\alpha \odot \beta \qquad \alpha \qquad \dfrac{[\beta]}{\alpha \odot \beta}\,\odot\text{-I}_1 \qquad \beta \qquad \dfrac{[\alpha]}{\alpha \odot \beta}\,\odot\text{-I}_2}{\alpha \odot \beta}\,\odot\text{-E}'$$

In the sub-derivation of $\alpha \odot \beta$ from β, α has to be discharged vacuously, as does β in the sub-derivation of $\alpha \odot \beta$ from α; and the re-introduction of $\alpha \odot \beta$ requires sub-proofs of α and β, which will generally not be possible.

It therefore seems that, in general, the higher-level rule is stronger than its flattened version. At least, this is clearly so in intuitionistic logic. Take the rules for the novel connective c_2 introduced in Schroeder-Heister (nd):

$$\dfrac{\begin{matrix}[\alpha_1]\\ \vdots \\ \alpha_2\end{matrix}}{c_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4)}\,c_2\text{-I}_1 \qquad \dfrac{\begin{matrix}[\alpha_3]\\ \vdots \\ \alpha_4\end{matrix}}{c_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4)}\,c_2\text{-I}_2$$

Considerations of GE-harmony yield a single higher-level E-rule with two minor premises:

$$\dfrac{c_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \qquad \begin{matrix}[\alpha_1 \Rightarrow \alpha_2]\\ \vdots \\ \gamma\end{matrix} \qquad \begin{matrix}[\alpha_3 \Rightarrow \alpha_4]\\ \vdots \\ \gamma\end{matrix}}{\gamma}\,c_2\text{-E}$$

and the corresponding flattened rule:

$$\dfrac{c_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \qquad \alpha_1 \qquad \begin{matrix}[\alpha_2]\\ \vdots \\ \gamma\end{matrix} \qquad \alpha_3 \qquad \begin{matrix}[\alpha_4]\\ \vdots \\ \gamma\end{matrix}}{\gamma}\,c_2\text{-E}'$$

Let $c_2(\vec{\alpha})$ abbreviate $c_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, and $\vee(\vec{\alpha})$ abbreviate $(\alpha_1 \to \alpha_2) \vee (\alpha_3 \to \alpha_4)$. With the higher-level rule, c_2-E, we can show by intuitionistically acceptable means that $c_2(\vec{\alpha}) \dashv\vdash (\alpha_1 \to \alpha_2) \vee (\alpha_3 \to \alpha_4)$:

$$\dfrac{c_2(\vec{\alpha}) \qquad \begin{array}{c}\dfrac{\dfrac{\dfrac{\overline{\alpha_1}^{\,(1)} \quad \overline{\alpha_1 \Rightarrow \alpha_2}^{\,(2)}}{\alpha_2}}{\alpha_1 \to \alpha_2}\ {\to}\text{-I}_1(1)}{} \\ \vee(\vec{\alpha})\end{array} \qquad \begin{array}{c}\dfrac{\dfrac{\dfrac{\overline{\alpha_3}^{\,(3)} \quad \overline{\alpha_3 \Rightarrow \alpha_4}^{\,(4)}}{\alpha_4}}{\alpha_3 \to \alpha_4}\ {\to}\text{-I}_2(3)}{} \\ \vee(\vec{\alpha})\end{array}}{(\alpha_1 \to \alpha_2) \vee (\alpha_3 \to \alpha_4)}\ c_2\text{-E}(2,4)$$

Conversely,

$$\dfrac{(\alpha_1 \to \alpha_2) \vee (\alpha_3 \to \alpha_4) \qquad \dfrac{\dfrac{\overline{\alpha_1}^{\,(1)} \quad \overline{\alpha_1 \to \alpha_2}^{\,(2)}}{\alpha_2}}{c_2(\vec{\alpha})}\ c_2\text{-I}(1) \qquad \dfrac{\dfrac{\overline{\alpha_3}^{\,(3)} \quad \overline{\alpha_3 \to \alpha_4}^{\,(4)}}{\alpha_4}}{c_2(\vec{\alpha})}\ c_2\text{-I}(3)}{c_2(\vec{\alpha})}\ \vee\text{-E}(2,4)$$

With the flattened rules, however, it is not possible to derive $(\alpha_1 \to \alpha_2) \vee (\alpha_3 \to \alpha_4)$ from $c_2(\vec{\alpha})$ using intuitionistically valid rules. But with classical *reductio*, $\mathcal{CR}$, it is possible:

$$\dfrac{\dfrac{c_2(\vec{\alpha}) \quad \dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{\overline{\neg\alpha_1}^{\,(2)} \quad \overline{\alpha_1}^{\,(3)}}{\alpha_2}\,\mathcal{V}}{\alpha_1 \to \alpha_2}\ {\to}\text{-I}(3)}{} \\ \vee(\vec{\alpha})\end{array} \quad \overline{\neg \vee (\vec{\alpha})}^{\,(1)}}{\alpha_1}\ \mathcal{CR}(2) \quad \begin{array}{c}\dfrac{\dfrac{\overline{\alpha_2}^{\,(4)}}{\alpha_1 \to \alpha_2}}{} \\ \vee(\vec{\alpha})\end{array} \quad \dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{\overline{\neg\alpha_3}^{\,(5)} \quad \overline{\alpha_3}^{\,(6)}}{\alpha_4}\,\mathcal{V}}{\alpha_3 \to \alpha_4}\ {\to}\text{-I}(6)}{} \\ \vee(\vec{\alpha})\end{array} \quad \overline{\neg \vee (\vec{\alpha})}^{\,(1)}}{\alpha_3}\ \mathcal{CR}(5) \quad \begin{array}{c}\dfrac{\dfrac{\overline{\alpha_4}^{\,(7)}}{\alpha_3 \to \alpha_4}}{} \\ \vee(\vec{\alpha})\end{array}}{\vee(\vec{\alpha})}\ c_2\text{-E}'(4,7) \quad \overline{\neg \vee (\vec{\alpha})}^{\,(1)}}{(\alpha_1 \to \alpha_2) \vee (\alpha_3 \to \alpha_4)}\ \mathcal{CR}(1)$$

There is more than the *consequentia mirabilis* role of $\mathcal{CR}$ in play here (that is, to infer α from a demonstration that $\neg\alpha$ leads to contradiction). There is also much use of K and W in the multiple and vacuous discharge of assumptions in $\mathcal{CR}$ and $\to$-I.

$\odot$ is a special case of c_2, with $\alpha_1 = \alpha_4$ and $\alpha_2 = \alpha_3$, so we have a proof using classical *reductio* with the flattened E-rule that $(\alpha \to \beta) \vee (\beta \to \alpha) \dashv\vdash \alpha \odot \beta$. The reason $\alpha \odot \beta$ is classically derivable, and that $c_2(\vec{\alpha}) \dashv\vdash (\alpha_1 \to \alpha_2) \vee (\alpha_3 \to \alpha_4)$ is that the classical negation rules yield the full classical theory of implication, as in the multiple-conclusion sequent calculus, and the classical left-implication sequent calculus rule $\to$-left is invertible, that is, if $\Gamma, \alpha \to \beta \Rightarrow \Delta$ is derivable, so are $\Gamma \Rightarrow \alpha, \Delta$ and $\Gamma, \beta \Rightarrow \Delta$.[15] The classical negation rules of natural deduction and sequent calculus allow single-conclusion systems to mimic multiple-conclusion (at least to this extent)[16] by "parking" the negations of the parametric succedent formulae as antecedent fomulae (i.e., assumptions). Consider the following multiple-

[15] See, e.g., Proposition 3.5.4 (vi) in Troelstra and Schwichtenberg (2000, p. 79).

[16] But see Murzi and Hjortland (2009).

conclusion sequent calculus proof that $c_2(\vec{\alpha}) \vdash \vee(\vec{\alpha})$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha_1 \Rightarrow \alpha_1}{\alpha_1 \Rightarrow \alpha_1, \alpha_2}}{\Rightarrow \alpha_1, \alpha_1 \to \alpha_2}}{\Rightarrow \alpha_1, \vee(\vec{\alpha})} \quad \dfrac{\dfrac{\dfrac{\alpha_2 \Rightarrow \alpha_2}{\alpha_2, \alpha \Rightarrow \alpha_2}}{\alpha_2 \Rightarrow \alpha \to \alpha_2}}{\alpha_2 \Rightarrow \vee(\vec{\alpha})} \quad \dfrac{\dfrac{\dfrac{\alpha_3 \Rightarrow \alpha_3}{\alpha_3 \Rightarrow \alpha_3, \alpha_4}}{\Rightarrow \alpha_3, \alpha_3 \to \alpha_4}}{\Rightarrow \alpha_3, \vee(\vec{\alpha})} \quad \dfrac{\dfrac{\dfrac{\alpha_4 \Rightarrow \alpha_4}{\alpha_4, \alpha_4 \Rightarrow \alpha_4}}{\alpha_4 \Rightarrow \alpha_4 \to \alpha_4}}{\alpha_4 \Rightarrow \vee(\vec{\alpha})}}{c_2(\vec{\alpha}) \Rightarrow \vee(\vec{\alpha})}\ c_2\text{-left}$$

The rule c_2-left used here reads:

$$\dfrac{\Gamma \Rightarrow \alpha_1, \Delta \quad \Gamma, \alpha_2 \Rightarrow \Delta \quad \Gamma \Rightarrow \alpha_3, \Delta \quad \Gamma, \alpha_4 \Rightarrow \Delta}{\Gamma, c_2(\vec{\alpha}) \Rightarrow \Delta}\ c_2\text{-left}$$

The rules for negation in Gentzen's LK, his classical sequent calculus, are:

$$\dfrac{\Gamma \Rightarrow \Delta, \alpha}{\neg\alpha, \Gamma \Rightarrow \Delta}\ \neg\text{-left} \qquad \dfrac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\alpha}\ \neg\text{-right}$$

With these rules, within the multiple-conclusion system LK we can establish the following derived rule:

$$\dfrac{\neg\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha}\ \mathcal{C} \qquad \text{with proof:} \qquad \dfrac{\dfrac{\neg\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\neg\alpha} \quad \dfrac{\dfrac{\alpha \Rightarrow \alpha}{\Rightarrow \alpha, \neg\alpha}}{\neg\neg\alpha \Rightarrow \alpha}}{\Gamma \Rightarrow \Delta, \alpha}\ \text{Cut}$$

If we now move to a single-conclusion sequent calculus, we can use $\mathcal{C}$ (with Δ empty)[17] to derive two further classical single-conclusion negation rules, $\mathcal{CT}$ (i.e., contraposition) and $\mathcal{CM}$ (i.e., *consequentia mirabilis*):

$$\dfrac{\Gamma, \neg\alpha \Rightarrow \beta}{\Gamma, \neg\beta \Rightarrow \alpha}\ \mathcal{CT} \qquad \text{with proof:} \qquad \dfrac{\dfrac{\Gamma, \neg\alpha \Rightarrow \beta}{\Gamma, \neg\beta, \neg\alpha \Rightarrow}\ \neg\text{-left}}{\Gamma, \neg\beta \Rightarrow \alpha}\ \mathcal{C}$$

$$\dfrac{\Gamma, \neg\alpha \Rightarrow \alpha}{\Gamma \Rightarrow \alpha}\ \mathcal{CM} \qquad \text{with proof:} \qquad \dfrac{\dfrac{\dfrac{\Gamma, \neg\alpha \Rightarrow \alpha}{\Gamma, \neg\alpha, \neg\alpha \Rightarrow}\ \neg\text{-left}}{\Gamma, \neg\alpha \Rightarrow}\ \text{W}}{\Gamma \Rightarrow \alpha}\ \mathcal{C}$$

Then we can establish $c_2(\vec{\alpha}) \vdash \vee(\vec{\alpha})$ in single-conclusion sequent calculus using $\mathcal{CT}$ and $\mathcal{CM}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha_1 \Rightarrow \alpha_1}{\neg\alpha_2, \alpha_1 \Rightarrow \alpha_1}}{\neg\alpha_1, \alpha_1 \Rightarrow \alpha_2}\ \mathcal{CT}}{\neg\alpha_1 \Rightarrow \alpha_1 \to \alpha_2}}{\neg\alpha_1 \Rightarrow \vee(\vec{\alpha})}}{\neg \vee (\vec{\alpha}) \Rightarrow \alpha_1}\ \mathcal{CT} \quad \dfrac{\dfrac{\dfrac{\alpha_2 \Rightarrow \alpha_2}{\alpha_2, \alpha_1 \Rightarrow \alpha_2}}{\alpha_2 \Rightarrow \alpha_1 \to \alpha_2}}{\alpha_2 \Rightarrow \vee(\vec{\alpha})} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha_3 \Rightarrow \alpha_3}{\neg\alpha_4, \alpha_3 \Rightarrow \alpha_3}}{\neg\alpha_3, \alpha_3 \Rightarrow \alpha_4}\ \mathcal{CT}}{\neg\alpha_3 \Rightarrow \alpha_3 \to \alpha_4}}{\neg\alpha_3 \Rightarrow \vee(\vec{\alpha})}}{\neg \vee (\vec{\alpha}) \Rightarrow \alpha_3}\ \mathcal{CT} \quad \dfrac{\dfrac{\dfrac{\alpha_4 \Rightarrow \alpha_4}{\alpha_4, \alpha_3 \Rightarrow \alpha_4}}{\alpha_4 \Rightarrow \alpha_3 \to \alpha_4}}{\alpha_4 \Rightarrow \vee(\vec{\alpha})}}{\neg \vee (\vec{\alpha}), c_2(\vec{\alpha}) \Rightarrow \vee(\vec{\alpha})}\ c_2\text{-left}}{c_2(\vec{\alpha}) \Rightarrow (\alpha_1 \to \alpha_2) \vee (\alpha_3 \to \alpha_4)}\ \mathcal{CM}$$

[17] If we require the succedent to be non-empty, we can capture "empty" succedent with an instance of $\bot$.

This proof exhibits essentially the same proof architecture as the earlier proof of the same result in classical natural deduction.

4 Negation and Inconsistency

Often, $\neg\alpha$ is treated by definition as $\alpha \to \bot$, where $\bot$ is governed solely by an elimination-rule, from $\bot$ infer anything:

$$\frac{\bot}{\alpha}\ \bot\text{E}$$

Gentzen (1932) treated '$\neg$' as primitive. As introduction-rule, he took *reductio ad absurdum*, as noted in §3:

$$\frac{\begin{matrix}[\alpha]\\ \vdots\\ \beta\end{matrix}\qquad\qquad \begin{matrix}[\alpha]\\ \vdots\\ \neg\beta\end{matrix}}{\neg\alpha}\ \mathcal{R}$$

What elimination-rule does this justify? We can infer from $\neg\alpha$ whatever (all and only that which) we can infer from its grounds. There is one I-rule with two premises ($m = 1, n_1 = 2$), so there will be two E-rules, one for each premise of the I-rule:

$$\frac{\neg\alpha \qquad \begin{matrix}\left[\begin{matrix}[\alpha]\\ \vdots\\ \beta\end{matrix}\right]\\ \vdots\\ \gamma\end{matrix}}{\gamma}\ \neg\text{E}_1 \qquad\text{and}\qquad \frac{\neg\alpha \qquad \begin{matrix}\left[\begin{matrix}[\alpha]\\ \vdots\\ \neg\beta\end{matrix}\right]\\ \vdots\\ \gamma\end{matrix}}{\gamma}\ \neg\text{E}_2$$

Flattening the rules as before, where we infer γ from assuming the existence of derivations, respectively, of β and of $\neg\beta$ from α, we obtain:

$$\frac{\neg\alpha \qquad \alpha \qquad \begin{matrix}[\beta]\\ \vdots\\ \gamma\end{matrix}}{\gamma} \qquad \text{and so} \qquad \begin{matrix}\dfrac{\neg\alpha \qquad \alpha}{\beta}\\ \vdots\\ \gamma\end{matrix}$$

and

$$\frac{\neg\alpha \qquad \alpha \qquad \begin{matrix}[\neg\beta]\\ \vdots\\ \gamma\end{matrix}}{\gamma} \qquad \text{and so} \qquad \begin{matrix}\dfrac{\neg\alpha \qquad \alpha}{\neg\beta}\\ \vdots\\ \gamma\end{matrix}$$

The second of these is simply a special case of the first, and so we have justified Gentzen's form of *Ex Falso Quodlibet* as the matching E-rule for '$\neg$':[18]

$$\frac{\neg\alpha \qquad\qquad \alpha}{\beta}\ \mathcal{V}$$

The account of negation given by $\mathcal{R}$ and $\mathcal{V}$ is intuitionistic. But similar arguments extend this account to classical negation, by setting it in a multiple-conclusion framework.[19] $\mathcal{R}$ generalizes to a multiple-conclusion rule as:

$$\frac{\begin{array}{c}[\alpha]\\ \vdots\\ \beta,\Delta\end{array} \qquad\qquad \begin{array}{c}[\alpha]\\ \vdots\\ \neg\beta,\Delta\end{array}}{\neg\alpha,\Delta}\ \mathcal{R}_m$$

from which the inversion principle yields the pair of higher-order E-rules:

$$\frac{\neg\alpha,\Gamma \qquad \begin{array}{c}[\alpha\Rightarrow\beta]\\ \vdots\\ \Delta\end{array}}{\Gamma,\Delta} \qquad \text{and} \qquad \frac{\neg\alpha,\Gamma \qquad \begin{array}{c}[\alpha\Rightarrow\neg\beta]\\ \vdots\\ \Delta\end{array}}{\Gamma,\Delta}$$

which flatten and simplify as before to

$$\frac{\neg\alpha,\Gamma \qquad\qquad \alpha,\Delta}{\Gamma,\Delta}\ \mathcal{V}_m$$

From $\mathcal{R}_m$ and $\mathcal{V}_m$ we can derive double-negation elimination, and so justify $\mathcal{CR}$, as derived rules:

$$\frac{\neg\neg\alpha \qquad \dfrac{\dfrac{\overline{\ \alpha\ }^{\,(1)}}{\alpha,\alpha}\,\text{K} \qquad \dfrac{\overline{\ \alpha\ }^{\,(1)}}{\neg\alpha,\alpha}\,\text{K}}{\neg\alpha,\alpha}\,\mathcal{R}_m(1)}{\alpha}\ \mathcal{V}_m$$

[18] Recall from §2 Dummett's minimal I-rule for '$\neg$':

$$\frac{\alpha\Rightarrow\neg\alpha}{\neg\alpha}\ \neg\text{-I}$$

GE-considerations justify as E-rule:

$$\frac{\neg\alpha \qquad \begin{array}{c}[\alpha\Rightarrow\neg\alpha]\\ \vdots\\ \gamma\end{array}}{\gamma}$$

which flattens to

$$\frac{\neg\alpha \qquad \alpha}{\neg\alpha}$$

But this adds nothing to what was already derivable using $\neg$-I.

[19] See Read (2000, pp. 149-50).

Finally, consider the one-place connective, $\bullet$, whose single introduction-rule has one hypothetical premise:

$$\dfrac{\begin{matrix}[\bullet\alpha]\\ \vdots\\ \alpha\end{matrix}}{\bullet\,\alpha}\;\bullet\text{-I}$$

GE-harmony yields as E-rule in the usual way:

$$\dfrac{\bullet\alpha \qquad \begin{matrix}[\bullet\alpha \Rightarrow \alpha]\\ \vdots\\ \gamma\end{matrix}}{\gamma}\;\bullet\text{-E} \qquad \text{which flattens to} \qquad \dfrac{\bullet\alpha \qquad \bullet\alpha}{\alpha}\;\bullet\text{-E}$$

where $\bullet\alpha$ is both major and minor premise. $\bullet$ satisfies the inversion principle, whereby

$$\dfrac{\dfrac{\begin{matrix}[\bullet\alpha]\\ \mathcal{D}\\ \alpha\end{matrix}}{\bullet\alpha}\;\bullet\text{-I} \qquad \begin{matrix}\mathcal{D}'\\ \bullet\alpha\end{matrix}}{\alpha}\;\bullet\text{-E} \qquad \text{converts to} \qquad \begin{matrix}\mathcal{D}'\\ \bullet\alpha\\ \mathcal{D}\\ \alpha\end{matrix}$$

$\bullet\alpha$ is a formal Curry paradox, for $\bullet$ introduces inconsistency, in fact, triviality, since we can prove α, for any α:

$$\dfrac{\dfrac{\dfrac{\overline{\bullet\,\alpha}^{\,1} \qquad \overline{\bullet\,\alpha}^{\,1}}{\alpha}\;\bullet\text{-E}}{\bullet\,\alpha}\;\bullet\text{-I [1]} \qquad \dfrac{\dfrac{\overline{\bullet\,\alpha}^{\,2} \qquad \overline{\bullet\,\alpha}^{\,2}}{\alpha}\;\bullet\text{-E}}{\bullet\,\alpha}\;\bullet\text{-I [2]}}{\alpha}\;\bullet\text{-E}$$

(Note, however, the use of Contraction in each application of $\bullet$-I.) The proof fails to normalize, since clearly, if we try to remove the maximum formula $\bullet\alpha$ in the left-hand premise of the final use of $\bullet$-E, we obtain just the same proof again. How can we prevent this? Should it be prevented?

One proposal is Dummett's complexity constraint:

> "The minimal demand we should make on an introduction rule intended to be self-justifying is that its form be such as to guarantee that, in any application of it, the conclusion will be of higher complexity than any of the premisses and than any discharged hypothesis. We may call this the complexity condition."

Although this rules out $\bullet$, and classical *reductio*, it also rules out apparently innocuous rules such as Gentzen's $\mathcal{R}$ above, and even Dummett's own $\neg$-I rule for minimal negation. The moral to draw is that GE-harmony is not designed to rule out anything, but to ensure that the E-rules add no more (and no less) to whatever meaning is given by the assertion-conditions encapsulated in the I-rule(s). The I-rule for $\bullet$ already shows its inconsistency, which in turn justifies $\bullet$-E. Harmony does not import inconsistency, but serves to make it transparent.

5 Conclusion

Michael Dummett introduced the notion of harmony in response to Arthur Prior's tonkish attack on the idea of proof-theoretic justification of logical laws (or analytic validity). Dummett developed the notion of harmony in different ways, in an attempt to use the idea to underpin his anti-realism. One of these ways (so-called "intrinsic harmony") drew on work by Dag Prawitz, in which he articulated an idea of Gerhard Gentzen's into a procedure based on Lorenzen's inversion principle whereby elimination-rules are in the appropriate sense functions of the corresponding introduction-rules. Roy Dyckhoff and Nissim Francez coined the term "general-elimination harmony" for the relationship created by this procedure. GE-harmony ensures that meaning is given solely, and hence transparently, by the assertion-conditions encapsulated in the I-rule(s), in such a way that the E-rule(s) add no more and no less to whatever meaning is given by the assertion-conditions encapsulated in the I-rule(s). The E-rules which result may be of higher level, permitting the discharge of rules as well as formulae. Such rules can be flattened to rules in which only formulae are discharged, but only in the context of classical *reductio*, or a multiple-conclusion format, can we be sure that the flattened rules will be equivalent to the higher-level rules. In general, the flattened rules are weaker than the I-rule warrants, and so are not in Dummett's term, stable.

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